Superpotentials of Lienard-Wiechert potentials in far fields. II. The relativistic case

1993 J. Phys. A: Math. Gen. 26 4431

(http://iopscience.iop.org/0305-4470/26/17/047)

J. Phys. A: Math. Gen. 26 (1993) 4431–4439. Printed in the UK

Superpotentials of Liénard–Wiechert potentials in far fields: II. The relativistic case

H Kawaguchi and T Honma

Department of Electrical Engineering, Faculty of Engineering, Hokkaido University, Kita 13, Nishi 8, Kita-ku, Sapporo Japan

Received 15 February 1993, in final form 16 April 1993

Abstract. The estimation method of the particle trajectory from the far electromagnetic fields which are produced by the particle, is presented. Using the Liénard–Wiechert superpotentials, the authors have previously presented the estimation method of the particle trajectory for the case that the motion is periodic and non-relativistic. This paper indicates that the estimation method can be expanded to the relativistic case.

1. Introduction

One of the authors discussed superpotentials for the Liénard–Wiechert potentials (the Liénard-Wiechert superpotentials) [1]. The physical meaning of the superpotentials is 'the coordinate of the particle which produces the Liénard–Wiechert potentials'. This implies that if one can calculate the superpotentials from the electromagnetic fields, the particle trajectory can be estimated from the electromagnetic fields. The concrete estimation method of the source particle trajectory using this concept was presented in [2]. However, the estimation method presented was for the special case that the motion is periodic and non-relativistic. Also, one had to select appropriate observation points of the electromagnetic fields (this condition was not stated explicitly in [2]).

In this paper, a more general estimation method of the source particle trajectory from the electromagnetic fields is presented. That is to say, this method can be used for the relativistic case and the selection of observation points is not necessary.

2. Liénard–Wiechert superpotentials and far electromagnetic fields

In this section, the Liénard–Wiechert potentials and the relation between the Liénard–Wiechert superpotentials and the far electromagnetic fields are summarized.

The Liénard–Wiechert potentials A^i are expressed in the covariant form as

$$A^i(ct, x) = \frac{e}{4\pi\varepsilon_0 c^2} \frac{u^i(\tau)}{R_k(\tau)u^k(\tau)} \tag{1}$$

where $u^i = (\gamma c, \gamma v(t))$ is the four velocity of the particle, $\gamma = (1 - v^2/c^2)^{-1/2}$, e is elementary charge, ε_0 is a dielectric constant, c is the velocity of light and $R_k(\tau)$ is a displacement vector defined by $R_k(\tau)=x^i-y^i(\tau)$ (where $x^i(=(ct,x))$ is the four-dimensional coordinate of the observation point, $y^i(\tau)(=(c\tau,y(\tau)))$ is the four-dimensional position vector of the particle and $y(t)$ is a trajectory of the particle with a parameter t). Then, τ is the so-called 'retarded time' which satisfies the following causal relation:

$$\tau=t-\frac{|x-y(\tau)|}{c}. \tag{2}$$

Here, $\Box$ is D'Alembertian. Solving (2), one can regard the retarded time τ or $y^i(\tau)$ as a function of t and $x(\tau=\tau(ct,x)$ or $y^i=y^i[\tau(ct,x)])$. Then, using the function $y^i(\tau)$ (the Liénard–Wiechert superpotentials), the Liénard–Wiechert potentials can be expressed as [1]

$$A^i(ct,x)=-\frac{e}{8\pi\varepsilon_0 c}\Box\, y^i[\tau(ct,x)]. \tag{3}$$

And also, evaluating (3) in the Fourier form, the relation between the far electromagnetic fields and the superpotentials can be obtained as follows. That is to say, if the motion is periodic, the spatial component of the Liénard–Wiechert superpotentials $y(\tau)$ in the far fields are expressed as [2]

$$y(ct,x)=y_0(x)+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_a^b \sin\left[n\left(\omega t-\sigma-\frac{\omega}{c}|x-y(\sigma)|\right)\right]\mathrm{d}y(\sigma) \tag{4}$$

where

$$b(=a+2\pi)\equiv 2\pi-\frac{|x-y(b)|}{c} \tag{5}$$

$$y_0(x)=M\times\frac{x}{|x|} \tag{6}$$

$$M\equiv\frac{1}{4\pi}\frac{\omega}{c}\int_a^b y(\sigma)\times dy(\sigma) \tag{7}$$

and ω is the angular frequency of the periodic motion.

On the other hand, using the expression of the Liénard–Wiechert potentials

$$A^i(ct,x)=\frac{e\omega}{8\pi^2\varepsilon_0 c^2}\sum_{n=0}^{\infty}C_n\int_a^b\frac{\cos\left[n\left(\omega t-\sigma-\frac{\omega}{c}|x-y(\sigma)|\right)\right]}{|x-y(\sigma)|}\mathrm{d}y^i(\sigma) \tag{8}$$

where

$$C_n = \begin{cases} 1 & \text{for } n=0 \\ 2 & \text{for } n \neq 0 \end{cases} \tag{9}$$

the far electromagnetic fields can be expressed as

$$\begin{aligned} \boldsymbol{E}(ct, \boldsymbol{x}) = -\frac{e\omega^2}{4\pi^2\varepsilon_0 c^2}\frac{1}{|\boldsymbol{x}|}\sum_{n=1}^{\infty} n\Bigg[\frac{c}{\omega}\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\int_a^b \mathrm{d}\sigma \sin\left[n\left(\omega t - \sigma - \frac{\omega}{c}|\boldsymbol{x} - \boldsymbol{y}(\sigma)|\right)\right] \\ - \int_a^b \mathrm{d}\sigma \frac{\mathrm{d}\boldsymbol{y}(\sigma)}{\mathrm{d}\sigma}\sin\left[n\left(\omega t - \sigma - \frac{\omega}{c}|\boldsymbol{x} - \boldsymbol{y}(\sigma)|\right)\right]\Bigg] \end{aligned} \tag{10}$$

$$\begin{aligned} \boldsymbol{B}(ct, \boldsymbol{x}) = \frac{e\omega^2}{4\pi^2\varepsilon_0 c^2}\frac{1}{|\boldsymbol{x}|}\sum_{n=1}^{\infty}\frac{n}{c}\frac{\boldsymbol{x}}{|\boldsymbol{x}|} \\ \times \int_a^b \mathrm{d}\sigma \frac{\mathrm{d}\boldsymbol{y}(\sigma)}{\mathrm{d}\sigma}\sin\left[n\left(\omega t - \sigma - \frac{\omega}{c}|\boldsymbol{x} - \boldsymbol{y}(\sigma)|\right)\right]. \end{aligned} \tag{11}$$

Now, introducing the following notation:

$$\begin{aligned} f_n(ct, \boldsymbol{x}) \equiv \int_a^b \mathrm{d}\sigma \sin\left[n\left(\omega t - \sigma - \frac{\omega}{c}|\boldsymbol{x} - \boldsymbol{y}(\sigma)|\right)\right] \\ \boldsymbol{g}_n(ct, \boldsymbol{x}) \equiv \int_a^b \mathrm{d}\sigma \frac{\mathrm{d}\boldsymbol{y}(\sigma)}{\mathrm{d}\sigma}\sin\left[n\left(\omega t - \sigma - \frac{\omega}{c}|\boldsymbol{x} - \boldsymbol{y}(\sigma)|\right)\right] \end{aligned} \tag{12}$$

equations (10) and (11) are rewritten as

$$\boldsymbol{E}(ct, \boldsymbol{x}) = -\frac{e\omega^2}{4\pi^2\varepsilon_0 c^2}\frac{1}{|\boldsymbol{x}|}\sum_{n=1}^{\infty} n\left[\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\frac{c}{\omega}f_n(ct, \boldsymbol{x}) - \boldsymbol{g}_n(ct, \boldsymbol{x})\right] \tag{10a}$$

$$\boldsymbol{B}(ct, \boldsymbol{x}) = \frac{e\omega^2}{4\pi^2\varepsilon_0 c^2}\frac{1}{|\boldsymbol{x}|}\sum_{n=1}^{\infty}\frac{n}{c}\frac{\boldsymbol{x}}{|\boldsymbol{x}|} \times \boldsymbol{g}_n(ct, \boldsymbol{x}). \tag{11a}$$

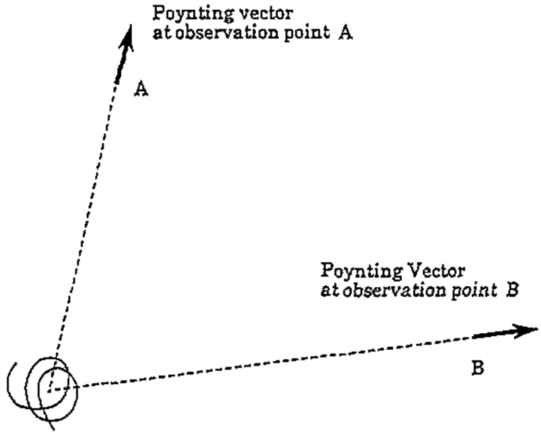

Figure 1. Estimation of the centre of trajectory from the Poynting vectors at two observation points.

Noting that

$$E = cB \times \frac{x}{|x|} \tag{13}$$

and using (10*a*) and (11*a*), we obtain

$$g_n(ct, x) \cdot \frac{x}{|x|} = \frac{c}{\omega} f_n(ct, x). \tag{14}$$

This implies

$$E(ct, x) = -\frac{e\omega^2}{4\pi^2\varepsilon_0 c^2}\frac{1}{|x|}\sum_{n=1}^{\infty} n\left[\frac{x}{|x|}\left(g_n(ct, x)\cdot\frac{x}{|x|}\right) - g_n(ct, x)\right]$$

$$\equiv \frac{e\omega^2}{4\pi^2\varepsilon_0 c^2}\frac{1}{|x|}\sum_{n=1}^{\infty} nh_n(ct, x). \tag{10b}$$

It is clear that the vector h_n is the normal component of vector g_n to the unit vector $x/|x|$. Then, the Liénard–Wiechert superpotentials (4) are rewritten as

$$y(ct, x) = y_0(x) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n} g_n(ct, x). \tag{4a}$$

It should be noticed that the first term of the (4*a*) is independent of the time *t* and that the second term is periodic with respect to the time *t*. Therefore one can interpret the first term as the mean position in the particle trajectory. From here, we shall call the first term 'the centre of the particle trajectory'. It is noticed that the centre of the particle trajectory can be found from the electromagnetic fields (or the Poynting vectors) at any two points (figure 1). Also, the unit vector $x/|x|$ can be calculated using the centre of the coordinate and the observation point. Then, if one selects the appropriate observation points (i.e. $g_n \cdot x/|x| = 0$), we obtain

$$E(ct, x) = \frac{e\omega^2}{4\pi^2\varepsilon_0 c^2}\frac{1}{|x|}\sum_{n=1}^{\infty} n g_n(ct, x). \tag{15}$$

Therefore, the vector g_n (or the superpotentials y) can be calculated from each Fourier component of the far electromagnetic fields (E_n) directly (because the first term of the (4*a*) can be neglected in the non-relativistic case). In [2], this case was considered. However, one can find the estimation method for the more general case as follows. After finding the centre of the trajectory from the Poynting vectors, one can establish the coordinate system appropriately. Then, if one observes the electromagnetic fields on the axis (i.e. $x/|x| = i, j, k$ where i, j and k are the unit vectors of the coordinate system), the projected vectors h_n of the vectors g_n on the y–z, z–x and x–y plane are obtained (figure 2). Now, it is found from (4*a*) that the following relation between the superpotentials and the electromagnetic fields is derived:

$$\begin{aligned} y(ct, x) - \left(y(ct, x)\cdot\frac{x}{|x|}\right)\frac{x}{|x|} &= y_0(ct, x) - \left(y_0(ct, x)\cdot\frac{x}{|x|}\right)\frac{x}{|x|} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[g_n(ct, x) - \left(g_n(ct, x)\cdot\frac{x}{|x|}\right)\frac{x}{|x|}\right] \\ y_\perp(ct, x) &= y_0(ct, x) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}h_n(ct, x) \\ y_\perp(ct, x) &= y_0(ct, x) + \frac{4\pi\varepsilon_0 c^2}{e\omega^2}|x|\sum_{n=1}^{\infty}\frac{1}{n^2}E_n(ct, x) \end{aligned} \tag{16}$$

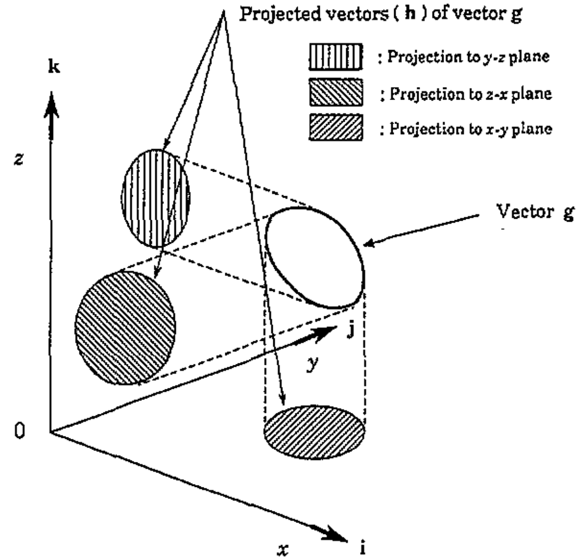

Figure 2. Projected vector h and its origin vector g.

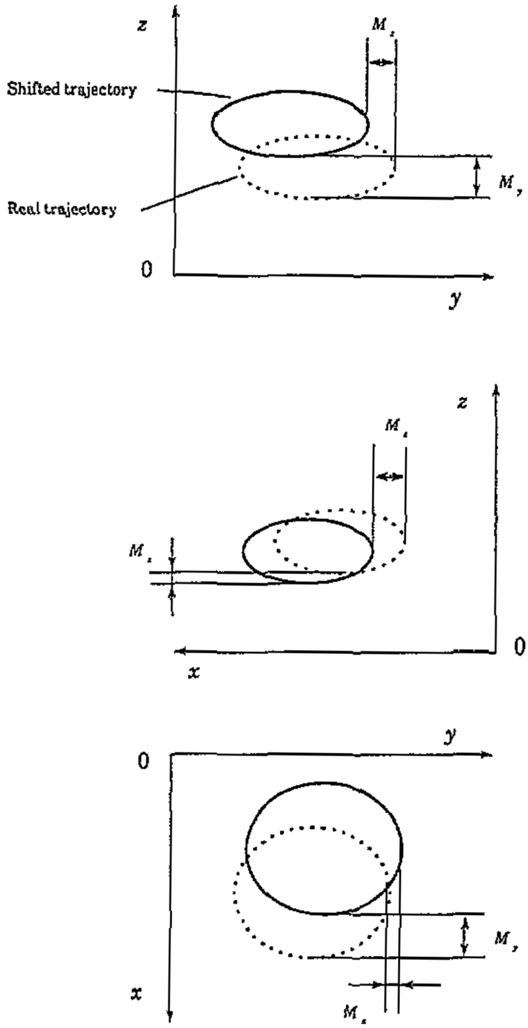

Figure 3. Shifted trajectories and real trajectories in each plane.

where $\boldsymbol{y}_\perp$ is the projected trajectory on the plane perpendicular to the vector $\boldsymbol{x}/|\boldsymbol{x}|$. Therefore, when $\boldsymbol{x}/|\boldsymbol{x}| = \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, one can calculate the projected trajectories $\boldsymbol{y}_\perp$ on the each plane which are shifted by the vector $\boldsymbol{y}_0 (= \boldsymbol{M} \times \boldsymbol{x}/|\boldsymbol{x}|)$ using the electromagnetic fields. Here, it should be noticed that for the above three observation points $(\boldsymbol{x}/|\boldsymbol{x}| = \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, there are the following expression for the vectors $\boldsymbol{y}_0$:

$$\boldsymbol{M} \times \boldsymbol{i} = (0, M_z, -M_y) \tag{17a}$$

$$\boldsymbol{M} \times \boldsymbol{j} = (-M_z, 0, M_x) \tag{17b}$$

$$\boldsymbol{M} \times \boldsymbol{k} = (M_y, -M_x, 0). \tag{17c}$$

Therefore, one can calculate the each components of the vector $\boldsymbol{M}$ from the relative

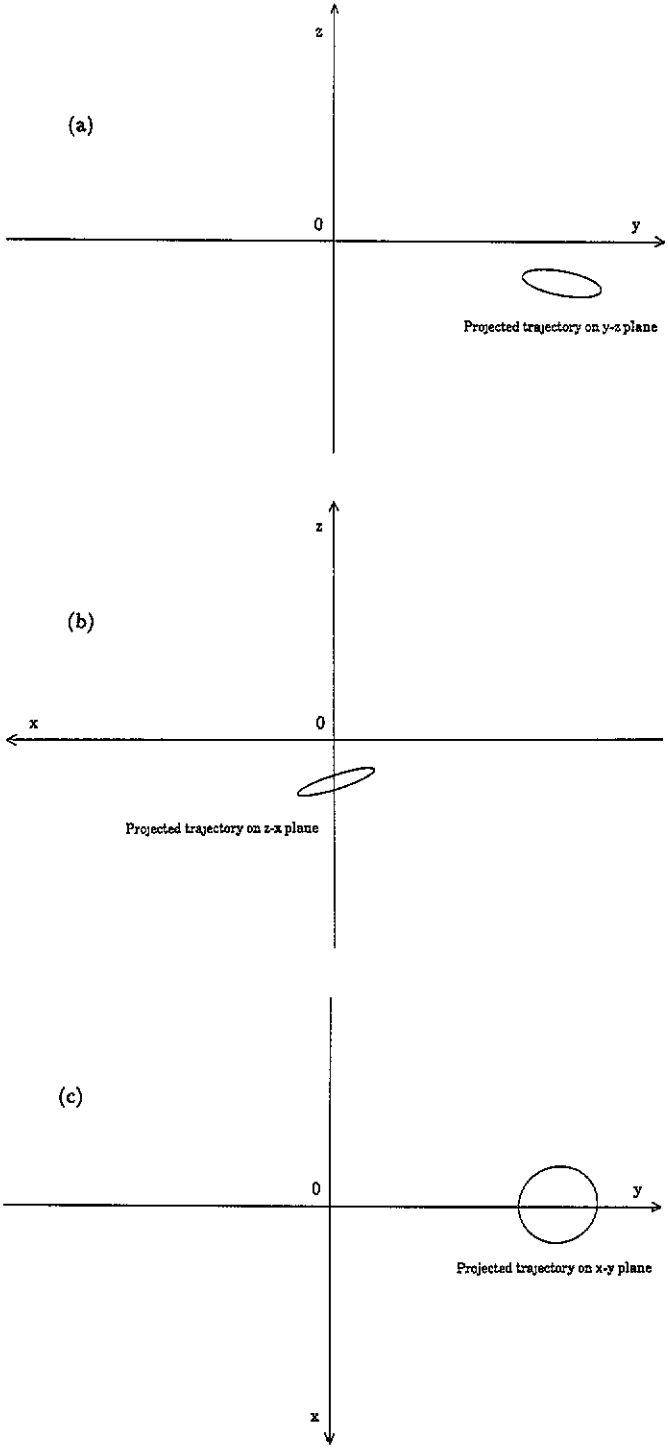

Figure 4. (a) The projection on the y–z plane of the particle trajectory for circular motion; (b) the projection on the z–x plane of the particle trajectory for circular motion; (c) the projection on the x–y plane of the particle trajectory for circular motion.

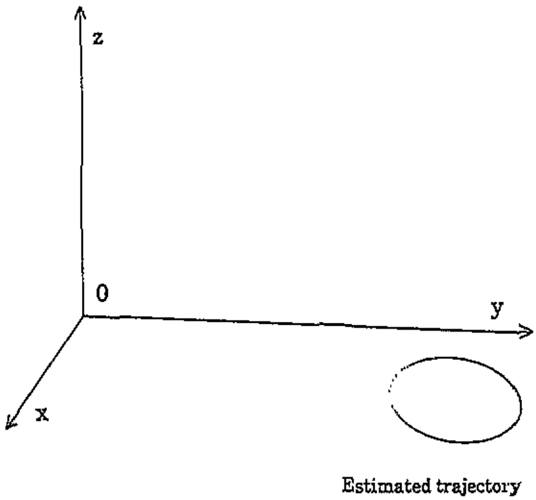

Figure 5. The estimated trajectory from the projections for circular motion.

shifts of the shifted trajectories, $y_{\perp} - y_0$ (figure 3), solving the following equations:

$$\begin{aligned} M_y + M_x &= A \\ M_z + M_x &= B \\ M_z + M_y &= C \end{aligned} \tag{18}$$

where A is the relative shift between the projected trajectories in the y–z plane and in the z–x plane for the z direction, B is the relative shift between the projected trajectories in the x–y and y–z planes for the y direction and C is the relative shift between the projections in the z–x and x–y plane for the x direction. If the vector $\boldsymbol{M}$ is obtained from (18), the vector y_0 can be easily calculated using (6). Then, using the

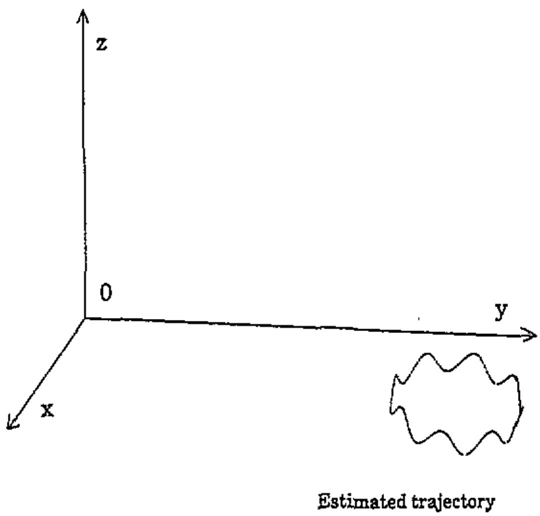

Figure 6. The estimated trajectory from the projections for undulating motion.

vector y_0, non-shifted projected trajectories $y_\perp$ are obtained. Finally, the real trajectory y can be estimated from the projected trajectories $y_\perp$ on each plane.

From the above discussions, one can say that the Liénard–Wiechert superpotentials $y(ct, x)$ or the particle trajectory can be calculated from the far electromagnetic fields which are produced by the particle, even if the particle motion is relativistic.

3. Numerical simulations

In this section, the estimation method presented in the previous section is confirmed using numerical simulations, then a caveat on the implementation of the method is stated.

First, circular motion of the particle in the constant magnetic field is considered, but, in this case, the orientation of the magnetic field does not coincide with any axes.

Here, there is a caveat on the calculation of the centre of the trajectory. In the relativistic case, the origin of the Poynting vector does not coincide with the centre of the trajectory, because most of the radiated power is emitted from a very small part of the trajectory [3]. Therefore, one should use Poynting vectors which are observed at a large number of observation points and take the mean value of these centre points.

In figures 4(a), 4(b) and 4(c), the projections of the particle trajectories in the constant magnetic fields onto the y–z, z–x and x–y plane are drawn. In figure 5 the estimated trajectory from the projections is drawn. The velocity of the particle is 95% of the velocity of light.

In figure 6, as an example of a three-dimensional trajectory, an undulating trajectory is shown. The velocity of the particle is 95% of the velocity of light.

In each figure, the trajectories are estimated exactly.

4. Summary

In this paper, the relation between the particle trajectory and the far electromagnetic fields has been presented for any periodic motion. One finds that using this relation, the source particle trajectory can be estimated using the far electromagnetic fields, even if the motion is relativistic.

Acknowledgments

The authors would like to thank Dr H Igarashi for helpful encouragement.

References

[1] Kawaguchi H and Murata S 1989 *J. Phys. Soc. Jpn* **58** [3] 848
[2] Kawaguchi H and Honma T 1992 *J. Phys. A: Math. Gen.* **25** 4437
[3] Panofsky W K H and Phillips M 1961 *Classical Electricity and Magnetism* (Cambridge MA: Addison-Wesley) section 20.4